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An algorithm for ranking the nodes of multiplex networks with data based on the PageRank concept[☆]

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ABSTRACT

A new algorithm for attributed multiplex networks is proposed and analysed with the main objective to compute the centrality of the nodes based on the original PageRank model used to establish a ranking in the Web pages network. Taking as a basis the Adapted PageRank Algorithm for monoplex networks with data and the two-layer PageRank approach, an algorithm for biplex networks is designed with two main characteristics. First, it solves the drawback of the existence of isolated nodes in any of the layers. Second, the algorithm allows us to choose the value of the parameter α controlling the importance assigned to the network topology and the data associated to the nodes in the Adapted PageRank Algorithm, respectively. The proposed algorithm inherits this ability to determine the importance of node attribute data in the calculation of the centrality; yet, going further, it allows to choose different α values for each of the two layers. The biplex algorithm is then generalised to the case of multiple layers, that is, for multiplex networks. Its possibilities and characteristics are demonstrated using a dataset of aggregate origin-destination flows of private cars in Rome. This dataset is augmented with attribute data describing city locations. In particular, a biplex network is constructed by taking the data about car mobility for layer 1. Layer 2 is generated from data describing the local bus transport system. The algorithm establishes the most central locations in the city when these layers are intertwined with the location attributes in the biplex network. Four cases are evaluated and compared for different values of the parameter that modulates the importance of data in the network.

1. Introduction

Identifying influential vertices in a network can be useful in many practical fields. Examples of this are risk identification in infrastructure [1], determining influential nodes in social networks [2], ensuring the security and reliability of the network [3], collaborating with the most influential media for advertising [4,5], evaluating the influence of junctions with the aim of avoiding overloading roads [6,7], or defining the influence maximization problem as an algorithmic problem [8].

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Various centrality measures including closeness, degree, and betweenness centrality [9] are widely used to this end, with the choice of the measure depending on the specific application. Further, PageRank and related algorithms have been proposed, extending the concept of network centrality and the range of applications [10,11]. The main idea of the PageRank algorithm proposed by Page et al. [12] is that a network node is relevant if other important nodes have a link to it. If a node with a high PR value is linked to another node, the value of the page being linked increases. It plays a crucial role in the ranking of nodes in complex networks [13,14]. Many scientists have used the PageRank and its modifications to address various problems. In [15], Wu and Chen introduce a hierarchical hybrid ranking algorithm to study entrepreneurship and innovation activities. The relative importance of scientific articles based on PageRank is presented in Wang et al. [16]. A social activity ranking method based on the PageRank algorithm is introduced in Nguyen et al. [17]. Ma et al. [18] create a novel ImageRank algorithm for image retrieval and relevance feedback. In [19], the authors design an algorithm similar to PageRank to identify important news events.

One of the most important characteristics of complex systems is that the collective behaviour of the system cannot be predicted from the properties of its components. The many types of inter-dependencies call for new ways to represent networks in which nodes have more than one type of interaction between them. These systems –called multiplex networks– are characterized by different layers representing different interaction types between nodes. An overview of research on multiplex networks can be found in Boccaletti et al. [20], Kivela et al. [21]. Modelled by a set of networks with interacting layers, these multilayer networks have been used to describe a many real-world complex systems, such as financial [22], ecological [23], information [24], urban [25], and transportation networks [26]. The recent advances in Big Data technologies allow to capture more and more types of relations in observed systems. In this context, it may be advantageous to represent and study these systems by representing them by multiplex networks [27–29].

Multiplex networks allow to connect pairs of nodes with multiple links in multiple layers. In addition, the ability to capture relations *between* layers is also important in modelling and explaining empirical multilayer networks. Ranking the nodes of these multiplex networks requires to highlight the importance of nodes in each of the interdependent layers [30]. In [31], Halu et al. proposed a PageRank algorithm for measuring node centralities in multiplex networks by introducing a bias exerted by a network layer on the jumps of the random walk in another layer. However, in many real world networks, attributes described by data intrinsic to the nodes play an important role and require further modelling. In order to take into account the data intrinsic to nodes, in this paper, we present a network centrality measure for biplex networks. The proposed algorithm is based on the Adapted PageRank Algorithm (APA) centrality [10,11] and is further is extended in a natural way to multiplex networks. A key aspect of the proposed algorithm is that, built upon the APA centrality, the random walk jumps in the algorithm are modelled by the network node attribute data. This allows us to study a range of relationships between among nodes modelled by different layers, as well as to measure the influence of the node attribute data in each of the network layers. In addition, the proposed algorithm can be applied in any multilayer setting, since it avoids the problem of isolated nodes in any of the layers.

To achieve these objectives, the structure of this paper is as follows. A method to construct a multiplex centrality based is the APA model is presented in Section 2. In Section 3, the biplex centrality algorithm is modified with the aim to avoid the existence of dangling nodes. Then, some characteristics about the meaning of the parameter *alpha* are discussed (Section 4). Section 5 aims to show an extension of the centrality to multiplex networks. Numerical results after analysing a real urban network in Rome are presented in Section 6. Finally, the conclusions of the work are exposed.

2. Constructing the multiplex centrality based on the APA model

Some classical notation for graphs will be used. So, a graph is represented by $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with $\mathcal{N}$ a set of n vertices or nodes and $\mathcal{E}$ a set of links between the nodes. The links are represented by the adjacency matrix $A = \left(a_{ij}\right)$ square of size $n \times n$, where

$$a_{ij} = \begin{cases} 1 & \text{if it exists a link between the nodes } i \text{ and } j, \\ 0 & \text{otherwise.} \end{cases}$$

2.1. Previous work

This section briefly describes the steps that take us from the original system to the model proposed to measure the centrality of the nodes of a multiplex complex network.

PageRank is an algorithm, based on the webgraph, that produces a classification of the web pages according to their importance.

The core of PageRank is the construction of the called *Google matrix*

$$G_{ij} = \alpha S_{ij} + (1 - \alpha)\frac{1}{N}, \tag{1}$$

where S_{ij} is, by columns, an stochastic matrix obtained from the adjacency matrix of the graph and the number of outgoing links from a node to the rest. The existence of null columns in S is a consequence of isolated nodes, which is solved by introducing a constant value $1/N$ –with N the number of nodes–. The parameter labeled as α is known as the *damping factor*. As it is well-known, the spectral properties of G, defined by (1) causes Perron-Frobenius's theorem to be satisfied, so for $0 < \alpha < 1$, there exists a unique and non-negative eigenvector associated to the maximal eigenvalue $\lambda = 1$. This vector –called PageRank vector– constitutes the rank of the nodes.

In the APA model described by the Adapted PageRank Algorithm (see [10], page 2190), a similar reasoning is used to determine a ranking of the nodes according to their importance.

In this case, the core of the model is the construction of a matrix M_{APA} given by

$$M_{APA}(ij) = (1-\alpha)P_{ij} + \alpha V. \tag{2}$$

The *transition matrix* P_{ij} is defined as

$$p_{ij} = \begin{cases} \frac{1}{c_j} & \text{if } a_{ij} \neq 0, \\ 0 & \text{otherwise,} \end{cases} \quad 1 \leq i,\ j \leq n, \tag{3}$$

with c_k representing the sum of the kth column of the adjacency matrix. The matrix V in the second term of Eq. (2) is constructed from a data matrix associated to the network. For more details, see [10].

It is important to highlight that the M_{APA} matrix, due to the way it is defined by (2) and (3), inherits the spectral characteristics of the Google matrix, with the property of being a stochastic matrix by columns. This fact assures us the existence of the dominant eigenvalue $\lambda = 1$ and the consequent right-side eigenvector that constitutes the expected classification of the nodes.

The second idea in which it is based the proposed algorithm for multiplex networks is the PageRank approach described by Pedroche et al. [32], known as *two-layer approach*. They state that Google matrix (1) may be divided into two terms and associated to two different layers representing the network. On the one hand, the *physical layer*, given by the term αS, and, on the other side, a *teleportation layer*, given by the term $1/N$.

In mathematical terms, Pedroche et al. [32] construct the 2×2 block matrix

$$M_A = \begin{pmatrix} \alpha P_A & (1-\alpha)I \\ 2\alpha I & (1-\alpha)ev^T \end{pmatrix} \in \mathbb{R}^{2n\times 2n}. \tag{4}$$

where M_A represents a Markov chain.

The algebraic characteristics of M_A –irreducible and primitive– allows us to affirm that

$$\hat{\pi}_A = \pi_u + \pi_d \in \mathbb{R}^n,$$

where $\left[\pi_u^T\ \pi_d^T\right]^T \in \mathbb{R}^{2n}$, is the only positive and normalised eigenvector. The structure of this matrix will be generalised for the case of multiplex networks as it will be described in the following section.

In Fig. 1 an schematic graphic of the models used to design and implement the APA biplex algorithm for calculating the nodes' centrality in biplex networks is shown. For both detailed description –the APA algorithm and the two-layers PageRank approach–, see [10,25,32].

2.2. Constructing a biplex centrality from the APA algorithm and the two-layers approach

Taking into account that a multiplex networks is a case in which there are different relationships in each layer but the same nodes in all of them, we can extend the two layer approach to the case of multiplex netwoks. Behind this process is the idea of applying the two-layer model to every layer of the multiplex network.

Let us follow the classical notation for a multiplex network $\mathcal{M} = (\mathcal{N}, \mathcal{E}, \mathcal{S})$ with $\mathcal{S} = (l_1, l_2, \ldots, l_k)$ a set of layers.

Considering the simplest case of a biplex networks $\mathcal{M} = (\mathcal{N}, \mathcal{E}, \mathcal{S})$, with two layers $\mathcal{S} = (l_1, l_2)$ with adjacency matrices $A_1, A_2 \in R^{n\times n}$.

A matrix M_2 [32] is constructed as

$$M_2 = \frac{1}{2}\begin{pmatrix} \alpha P_1 & I & (1-\alpha)I & 0 \\ I & \alpha P_2 & 0 & (1-\alpha)I \\ 2\alpha I & 0 & (1-\alpha)ev_1^T & (1-\alpha)ev_2^T \\ 0 & 2\alpha I & (1-\alpha)ev_1^T & (1-\alpha)ev_2^T \end{pmatrix} \tag{5}$$

with P_i, for $i = 1, 2$ a stochastic matrices and v_i, for $i = 1, 2$ personalized vectors.

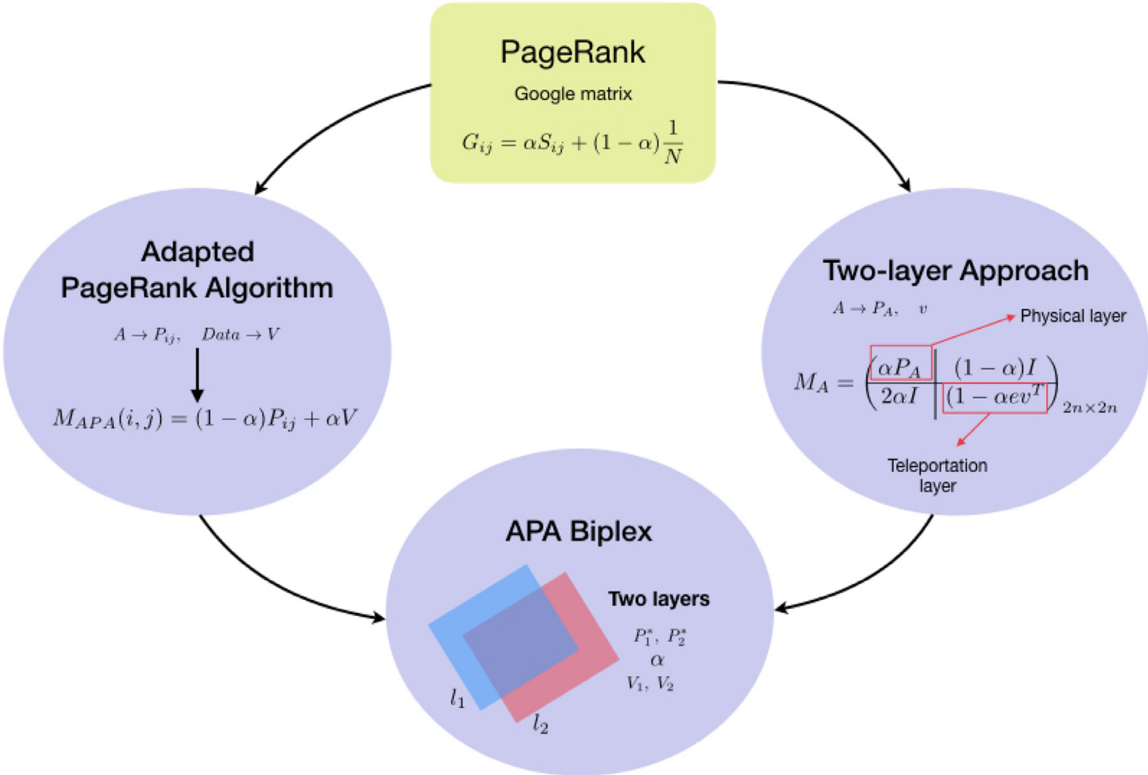
Thinking in terms of the APA algorithm, and taking the matrix M_{APA} (2) as the reference, a 2×2 block matrix is built

$$M_{APA2} = \begin{pmatrix} \alpha P_A & (1-\alpha)I \\ \alpha I & \alpha V \end{pmatrix} \in \mathbb{R}^{2n\times 2n}. \tag{6}$$

An algebraic study of the spectral characteristics of M_{APA2} allows us to affirm that this matrix is a class of Perron-Frobenius operators, ensuring the existence and uniqueness of the eigenvector associated with the dominant eigenvalue.

The following step consists on extending the two-layers APA model given by (6). Reordering the blocks so that the physical layer appear in the position (1,1) and the data layer in the block (2,2), we have

$$M_{BI} = \frac{1}{2}\begin{pmatrix} (1-\alpha)P_1 & I & 2(1-\alpha)I & 0 \\ I & (1-\alpha)P_2 & 0 & 2(1-\alpha)I \\ \alpha I & 0 & \alpha V_1 & \alpha V_2 \\ 0 & \alpha I & \alpha V_1 & \alpha V_2 \end{pmatrix} \tag{7}$$



$$M_{BI} = \frac{1}{2}\left(\begin{array}{cc|cc} (1-\alpha P_1^* & I & 2(1-\alpha)I & 0 \\ I & (1-\alpha P_2^* & 0 & 2(1-\alpha)I \\ \hline \alpha I & 0 & \alpha V_1 & \alpha V_2 \\ 0 & \alpha I & \alpha V_1 & \alpha V_2 \end{array}\right)_{4n\times 4n}$$

Fig. 1. Schematic representation of the models used to design the APA biplex centrality algorithm.

with P_1 and P_2 stochastic matrices by columns and V_i, for $i = 1, 2,$ data matrices of the two layers respectively.

As a consequencece, there exists an eigenvector

$$\hat{\pi}_{BI} = \left(\pi_{u_1}, \pi_{u_2}, \pi_{d_1}, \pi_{d_2}\right) \in \mathbb{R}^{4n} \tag{8}$$

associated to $\lambda = 1$ (largest eigenvalue).

This vector is essential and represents the node's centralities. Therefore, a unique vector can be obtained.

$$x = \frac{1}{2}\left(\pi_{u_1} + \pi_{u_2} + \pi_{d_1} + \pi_{d_2}\right) \in \mathbb{R}^n, \tag{9}$$

with x being a normalized vector.

2.3. Problems with dangling nodes in multiplex networks

Let us consider a biplex network with n nodes $\mathcal{N} = \{1, 2, \ldots, n\}$, and two layers, l_1 and l_2 corresponding to two different relationships between nodes that give rise to the adjacency matrices A_1 and A_2.

Because the nodes in the two layers are the same and the relationships are different, it is possible to for dangling nodes to appear in each of the layers; that is, nodes with no link to other nodes. In this way, we have nodes with degree greater than zero (*real nodes*) and nodes with degree equal to zero (*virtual nodes*). This has an undesirable effect on the spectral properties of the transition matrix P that is designed to be irreducible and stochastic by columns. The appearance of rows and columns of zeros corresponding to the virtual nodes causes P not to be stochastic, making the numerical resolution of the system based on the calculation of the PageRank vector impossible.

Let us discuss this instability with a simple case. Considering a biplex network with 10 nodes and the following two adjacency matrices

$$A_1 = \begin{pmatrix} 0&0&0&1&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0&0&0&0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0&0&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}. \quad (10)$$

We can observe from the adjacency matrices that in layer l_1 there is a dangling node (node 8) while in layer 2 there are two dangling nodes (nodes 2 and 10).

If we take A_1 from the first layer and calculate the transition matrix P_1, we notice that the sum vector of the columns is $sum = [2, 1, 2, 1, 1, 2, 2, 0, 2, 1]$. This leads us to a transition matrix

$$P_1 = \begin{pmatrix} 0&0&0&1&0&0&0&NaN&0&1\\ 0&0&1/2&0&0&0&0&NaN&0&0\\ 0&1&0&0&0&0&0&NaN&1/2&0\\ 1/2&0&0&0&0&0&0&NaN&0&0\\ 0&0&0&0&0&1/2&0&NaN&0&0\\ 0&0&0&0&1&0&1/2&NaN&0&0\\ 0&0&0&0&0&1/2&0&NaN&1/2&0\\ 0&0&0&0&0&0&0&NaN&0&0\\ 0&0&1/2&0&0&0&1/2&NaN&0&0\\ 1/2&0&0&0&0&0&0&NaN&0&0 \end{pmatrix},$$

where *NaN* is the acronym for *Not a Number*, the result of dividing by zero. As a consequence of this, the matrix

$$M_{APA} = (1-\alpha)P_1 + \alpha V_1$$

will not be stochastic by columns and it would be impossible to obtain a classification vector for the nodes.

If you look at layer l_2, the adjacency matrix A_2 has two rows of zeros, at nodes 2 and 10, which makes them dangling nodes. We deal with the same problem as in layer l_1 although now errors occur in the columns 2 and 10. The solution to this problem is addressed in the next section.

3. Adapting the biplex centrality based on the APA algorithm for dangling nodes

To solve the problem of isolated nodes, we must reformulate the basic principles of the PageRank model and the definition of the Google matrix for the most generic case. Let us consider the graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with dangling nodes, that is, nodes with zero degree. From the adjacency matrix $A = (a_{ij})$ of the graph G, the matrix $P = (p_{ij})$ is redefined as:

$$p_{ij} = \begin{cases} 0 & \text{if } i \text{ is a dangling node, for all} \quad j = 1, 2, \dots, n. \\ \frac{1}{c_j} & \text{if} \quad a_{ij} \neq 0. \end{cases} \quad (11)$$

In this case, with the aim to construct a column stochastic matrix, we substitute the matrix P in the calculation of the matrix M_{APA} with a new matrix P^* given by

$$P^* = P + d \cdot u^T, \quad (12)$$

where d is the vector characterizing the dangling nodes defined as

$$d_i = \begin{cases} 1, & \text{if } i \text{ is a dangling node,} \\ 0, & \text{otherwise,} \end{cases}$$

and $u \in \Re^n$ characterizes the distribution of dangling nodes such that $u > 0$ and $u^T e = 1$, with $e = (1, 1, \dots, 1)$.

Rewriting P as (12) we make sure that we continue working with a stochastic matrix by columns so that we have the proper spectral features to obtain the classification vector.

Another aspect that must be highlighted is related to the construction of the V matrix based on the individual data associated with each node of the graph. For those *virtual nodes* that have no connection with the rest of the nodes, the data associated to them is zero. That means that the data vector v has a zero component in all the positions corresponding to dangling nodes.

It may be the case of having a large number of virtual nodes, which would lead to a high number of null values in V. On the other hand, it may be convenient, not only from the point of view of numerical stability, but also from the idea of

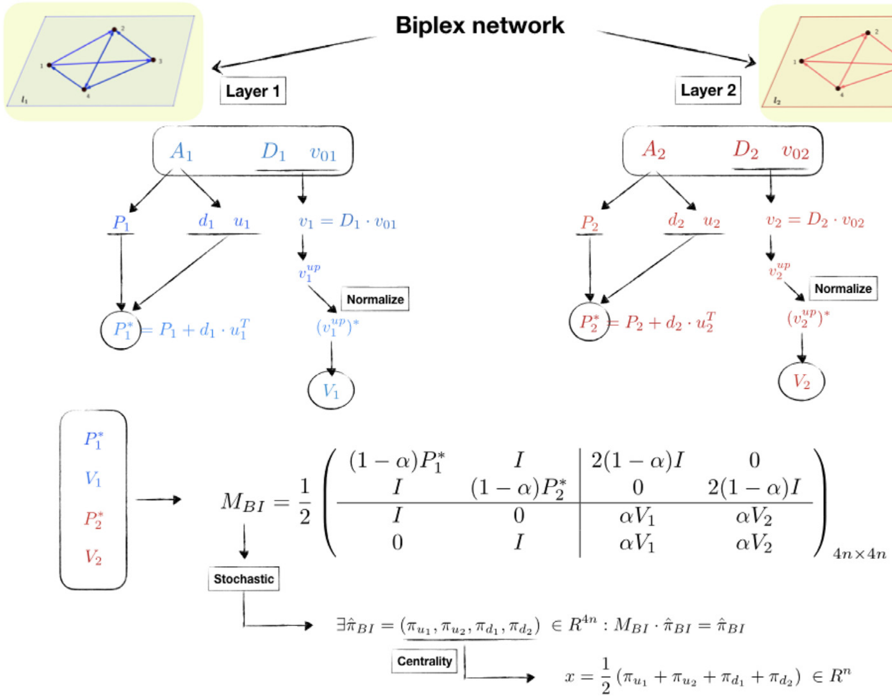


Fig. 2. Schematic representation of the APA biplex centrality model.

the influence of the whole data in the network, to introduce a small coefficient in places that have a null value of data. This small coefficient, that will be denoted as *coef*, may be defined as

$$coef = \frac{\min(D) > 0}{k - n}, \tag{13}$$

where D is the data matrix, k is the quantity of dangling nodes and n the number of nodes.

From this coefficient, it is possible to redefine the data vector v adding these small values at the positions of dangling nodes. This new data vector v^{up} is then given by

$$v^{up}(i) = \begin{cases} v(i), & \text{if } i \text{ has nonzero degree,} \\ coef, & \text{otherwise,} \end{cases}$$

As can be seen, the introduced coefficient represents a small value obtained by dividing the minimum data value associated to the nodes by the total number of dangling nodes.

Taking into account the modifications proposed regarding the matrix P^* and vector v^{up}, the matrix from which the centrality measure will be obtained may be rewritten as

$$M_{BI} = \frac{1}{2}\begin{pmatrix} (1-\alpha)P_1^* & I & 2(1-\alpha)I & 0 \\ I & (1-\alpha)P_2^* & 0 & 2(1-\alpha)I \\ \alpha I & 0 & \alpha V_1 & \alpha V_2 \\ 0 & \alpha I & \alpha V_1 & \alpha V_2 \end{pmatrix} \tag{14}$$

Fig. 2 details a graphic scheme of the centrality model for biplex networks –based on the APA algorithm– with dangling nodes. Note how the most notable changes with respect to the previous model occur in the construction of the matrices P_1^* and P_2^* as well as in the inclusion of a residual value in the data vector representing nodes with zero connectivity in any of the layers.

The scheme shown in Fig. 2 can be summarized in Algorithm 1 .

Following the simple example studied above with 10 nodes, where there were two layers with the adjacency matrices given by (10). Let us assume that the data matrices D_1, D_2 are

$$D_1 = [2, 2, 5, 2, 1, 3, 2, 0, 7, 2]^T, \quad D_2 = [4, 0, 5, 6, 1, 5, 2, 4, 3, 0]^T.$$

From the adjacency matrices, we detect the dangling nodes and, using the definitions of the vectors d_i and u_i, for $i = 1, 2$, we have that

$$d_1 = [0, 0, 0, 0, 0, 0, 0, 1, 0, 0]^T, \quad d_2 = [0, 1, 0, 0, 0, 0, 0, 0, 0, 1]^T.$$

Input: Let $\mathcal{M} = (\mathcal{N}, \mathcal{E}, \mathcal{S})$ be a biplex network with $\mathcal{N} = \{1, 2, \ldots, n\}$ the set of nodes, $\mathcal{S} = (l_1, l_2)$ two layers and A_1 and A_2 the respective adjacency matrices.
Let D_1 and D_2 the data matrices associated to nodes in layers l_1 and l_2, respectively, and weighted vectors $\vec{v_{01}}$ and $\vec{v_{02}}$, respectively.
Output: $\vec{x}$ representing the graph centrality
begin
For the layers l_i, for $i = 1, 2$, construct the vectors and matrices:
– P_i, the probability matrices from (3)
– vectors $\vec{d_i}, \vec{u_i}$ from the adjacency matrices A_i
Compute P_i^*, for $i = 1, 2$, from (12)
Compute the data vectors $\vec{v_i}$, for $i = 1, 2$, as $\vec{v_i} = D_i \cdot \vec{v_{0i}}$
Compute the coefficients $coef_i$, for $i = 1, 2$, from (8)
From $\vec{v_i}$ and $coef_i$, for $i = 1, 2$, compute $\vec{v}_i^{up}$
Normalize $\vec{v}_i^{up}$, for $i = 1, 2$ and denote it as $\{\vec{v}_i^{up}\}^*$
Construct V_i, for $i = 1, 2$
Construct the matrix M_{BI} from (14)
Compute the dominant eigenvector $\vec{\hat{\pi}}_{BI}$ of M_{BI}
Compute the centrality $\vec{x}$
end

Algorithm 1: APA biplex networks algorithm for computing the node's centrality.

$$u_1 = [0, 0, 0, 0, 0, 0, 0, 1, 0, 0]^T, \quad u_2 = [0, 1/2, 0, 0, 0, 0, 0, 0, 0, 1/2]^T.$$

Once P_i, d_i, u_i have been computed, the matrices P_i^* are obtained as

$$P_1^* = P_1 + d_1 \cdot u_1^T = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (15)$$

and

$$P_2^* = P_2 + d_2 \cdot u_2^T = \begin{pmatrix} 0 & 0 & 1/2 & 1 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix}. \quad (16)$$

Note that both P_i^* given by (15) and (16) are now column stochastic matrices.

Regarding the data vectors, we update vectors v_i to obtain vectors v_i^{up}, for $i = 1, 2$, by computing the following parameters involved in the updating process.

$$\min(D_1 > 0) = 1, \quad \min(D_2 > 0) = 1, \quad k_1 = 1, \quad k_2 = 2.$$

Using the expression (13), the respective coefficients are

$$coef_1 = 1, \quad coef_2 = 1/2.$$

In this way, the updated data vectors are

$$v_1^{up} = [2, 2, 5, 2, 1, 3, 2, 1, 7, 2]^T, \quad v_2^{up} = [4, 1/2, 5, 6, 1, 5, 2, 4, 3, 1/2]^T.$$

4. Some considerations about the α parameter

In the PageRank model, the parameter α represented the probability that a "surfer" follows the links of a web page uniformly and randomly. In this context, –of random surfer– the choice of α is not well justified, although in the original proposal of Page and Brin [33] they used $\alpha = 0.85$. In the extensive literature for web surfer, two choices are highlighted: $\alpha = 0.85$ and $\alpha = 0.5$ (see, for instance, [34]).

In the APA centrality algorithm, the α parameter has a different meaning, since it represents the importance we attach to the data associated with each node, while the value $(1-\alpha)$ represents the importance that we assign to the topology of the network we are studying.

In the design and assessment process of the different algorithms, both for single-layer networks and multi-layer networks, no mention about the alpha parameter at any time is done, since we assume that it is a fixed value that is initially chosen and used in both layers of the biplex network. This means that an equal importance to the data and the network connectivity in both layers is given. However, depending on the application context, it may happen that the node attribute data is more important in one of the layers and not in the other. If the same value of the parameter in the two layers is chosen, these differences will not be considered. The objective is to be able to set different parameter values for the different layers. We could consider the node attribute data to be more important than the topology in one layer but not the other. To contemplate these multiple possibilities, we need to introduce α_i, for layers l_i, with $i = 1, 2$.

Therefore, the matrix M_{BI} may be adapted as:

$$M_{BI} = \frac{1}{2}\begin{pmatrix} (1-\alpha_1)P_1^* & I & 2(1-\alpha_1)I & 0 \\ I & (1-\alpha_2)P_2^* & 0 & 2(1-\alpha_2)I \\ \alpha_1 I & 0 & \alpha_1 V_1 & \alpha_2 V_2 \\ 0 & \alpha_2 I & \alpha_1 V_1 & \alpha_2 V_2 \end{pmatrix} \tag{17}$$

and Algorithm 1 modified as:

Although the changes in the algorithms are minimal, the possibilities offered by being able to choose different alpha values in each layer are very important, since it allows to establish the importance that we associate to the node attribute data in each of the layers.

5. Extending the centrality to multiplex networks

Matrix M_{BI}, given by the expression 17, is the key in the entire process of the algorithm construction leading to the computation of the centrality. The spectral properties of this matrix and its stochastic characteristic allow us to calculate its dominant eigenvector that represents of the ranking of the nodes. A closer look at the structure of the matrix reveals that its extension to the case of multiple layers is easy. The block structure of this matrix favors its natural extension as we see below.

We have to remark that M_{BI} is constructed for biplex networks. Let us assume that we have a multiplex network with k layers $\{l_1, l_2, \ldots, l_k\}$, a set of adjacency matrices $\{A_1, A_2, \ldots, A_k\}$ and k data matrices $\{D_1, D_2, \ldots, D_k\}$. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the parameter values for the layers $l_1, l_2, \ldots, l_k$, respectively. Then, M_{BI} may be extended to multiplex networks as

$$M_{multi} = \frac{1}{k}\begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix} \tag{18}$$

with

$$M_{1,1} = \begin{pmatrix} (1-\alpha_1)P_1^* & I & \ldots & I \\ I & (1-\alpha_2)P_2^* & \ldots & I \\ \ldots & \ldots & \ldots & \ldots \\ I & I & \ldots & (1-\alpha_k)P_k^* \end{pmatrix}, \quad M_{2,2} = \begin{pmatrix} \alpha_1 V_1 & \alpha_2 V_2 & \ldots & \alpha_k V_k \\ \alpha_1 V_1 & \alpha_2 V_2 & \ldots & \alpha_k V_k \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_1 V_1 & \alpha_2 V_2 & \ldots & \alpha_k V_k \end{pmatrix}.$$

Both $M_{1,2}$ and $M_{2,1}$ are block diagonal matrices and are given by

$$M_{1,2} = \begin{pmatrix} 2(1-\alpha_1)I & 0 & \ldots & 0 \\ 0 & 2(1-\alpha_2)I & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 2(1-\alpha_k)I \end{pmatrix}, \quad M_{2,1} = \begin{pmatrix} \alpha_1 I & 0 & \ldots & 0 \\ 0 & \alpha_2 I & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \alpha_k I \end{pmatrix}.$$

The matrix M_{multi} constructed as shown in Eq. (18) inherits the spectral properties of M_{BI}; therefore, the dominant eigenvector allows us to compute the centrality x. We must also comment that the size of the matrix M_{multi} grows remarkably when we add layers in the network. More exactly, the size of M_{multi} is $2kn$, where k in the number of layers and n in the number of nodes. Thus, for large networks –with tens of thousands of nodes and more than 5 o 6 layers– it will be necessary to optimize the calculation of the dominant eigenvector.

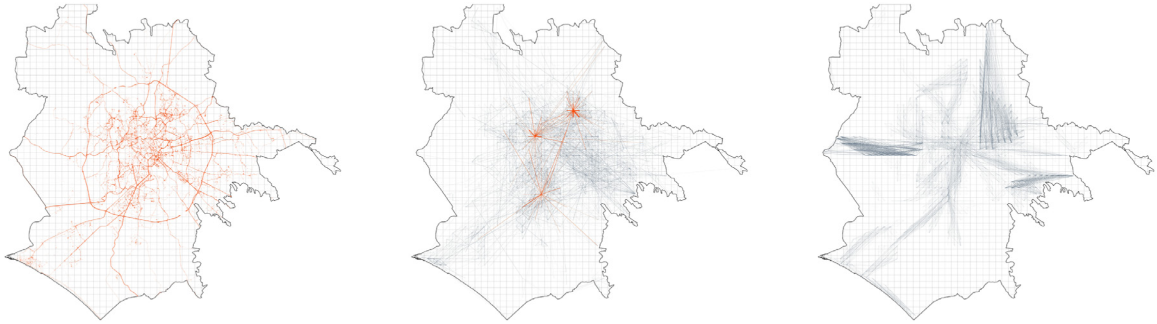


Fig. 3. (*left*) Private car GPS trajectories superimposed on the grid in Rome (*middle*) Layer 1 of biplex network: Rome OD network with some popular locations highlighted (*right*) Layer 2 of biplex network: bus connection network.

6. Numerical results

Nowadays, we show a real example of biplex network related to the urban network of the city of Rome, Italy. Data about car flows and the public bus transport system will be used to analyse and determine the most central areas of the city when both data are studied. To perform this, let us first briefly describe the dataset used for the numerical example.

6.1. Rome dataset

We use a custom dataset of aggregate origin-destination (OD) flows of private cars in Rome augmented with attribute data describing city locations in Rome. The workflow of building the dataset is as follows:

1. We subdivide the urban territory of Rome into n Cartesian grid cells of size 1×1 km, and consider each such quadratic cell as a node in the graph.
2. We have obtained GPS trajectory data of around 10,000 private cars spanning a period of one year from a car insurance company. After cleaning and processing the data, we superimpose the GPS trajectories on the urban grid (Fig. 3), and identify trip origins and destinations by setting a time threshold of the engine being off. Thresholds in the interval from 10 to 35 minyielded robust outcomes, and so we take 20 min as the threshold. We then map the identified origin-destination points to the grid cell which they fall into.
3. We build the first layer in the biplex network - the OD network - from the extracted origin-destination pairs by aggregating the flow counts over a year.
4. Similarly, we build the second layer of the biplex network by drawing an edge between a pair of grid cells if there is a public transport connection (bus) between them, and weight these edges by the number of such connections (Fig. 3).

6.2. The numerical results

As described in Section 6.1, we build a biplex network for the city of Rome with the nodes being the centroids of the grid representing the urban streets network. The two layers of the biplex network will represent the car flows between nodes and the bus transport system, respectively. As the biplex model allows us to establish different relationships between the nodes using different data, the idea in this case study is to analyse and visualise the relationship between a public transport system such as the urban bus connectivity and the car OD flows between different city locations.

Consequently, two layers may be defined with these characteristics:

- **Layer 1**: the graph is composed of the nodes of the urban network and an edge is drawn between two nodes if there is at least one car unit flow between these nodes. The attribute data associated to every node is the total quantity of in- and out-flows from the node.
- **Layer 2**: this layer graph has the same nodes, and two nodes are linked by an edge if there exists non-zero car flow between them and there also exists at least a bus line connecting them. The data associated to the nodes is the total number of bus lines connecting a node with the remaining ones.
  For instance, node number 9 is linked by at least one OD car flow with the nodes 1, 298, 416, 633, 713, 715, 730, 775, 999, 1083, 1087, and 1486. However, in layer 2 node 9 is linked only with the nodes 1 and 416 since there exists at least one bus line connecting the nodes. More precisely, between nodes 9 and 1 there are 6 lines connecting them and there are 2 bus lines connecting the nodes 9 and 416. Therefore, the data associated to the node 9 is $6 + 2 = 8$.

In this example we are quantifying and analysing urban mobility as well as the bus transport system. The advantage to work with two or more layers is that it is possible to measure several relationships with different datasets between nodes, which is not possible in networks with only one layer.

Another characteristic of this model is the possibility of differentiating the importance assigned to the data in each of the layers. In the definition of matrix M_{BI} given by the expression (17) each of the blocks has its own parameter α_i. This allows us to consider giving more importance to the data associated to the nodes in the first layer or, on the contrary, giving more value to the data in the second layer.

Case 1: $\alpha_1 = 0.2, \alpha_2 = 0.2$ In the first case analysed, we choose the same value of the parameter for both layers, that is, $\alpha_i = 0.2$, for $i = 1, 2$. This choice of parameters means that we are giving the same importance to data in both layers. Note that, following the original APA algorithm, the parameter varies in the interval (0,1); therefore, a low value of α_i also means that we give much more importance to the network topology than to the node attribute data. On the contrary, if we give a high value for the parameter, we are giving much more importance to the data associated to the nodes than to the graph topology.

We executed Algorithm 2 to obtain the biplex centrality of all the nodes according to the model described in this paper. In Fig. 4 (top) we show the map of Rome with the values of the biplex centrality, that we will denote in the following as PGBI, representing the size of the nodes proportional to their centrality value.

Input: Let $\mathcal{M} = (\mathcal{N}, \mathcal{E}, \mathcal{S})$ be a biplex network with $\mathcal{N} = \{1, 2, \dots, n\}$ the set of nodes, $\mathcal{S} = (l_1, l_2)$ two layers and A_1 and A_2 the respective adjacency matrices.
Let α_1 and α_2 be the values of the parameter α for layers l_1 and l_2, respectively. Let D_1 and D_2 be the data matrices associated to nodes in layers l_1 and l_2, respectively, and weighted vectors $\vec{v_{01}}$ and $\vec{v_{02}}$, respectively.
Output: $\vec{x}$ representing the graph centrality
begin
For the layers l_i, for $i = 1, 2$, construct the vectors and matrices:
– P_i, the probability matrices from (3)
– vectors $\vec{d_i}, \vec{u_i}$ from the adjacency matrices A_i
Compute P_i^*, for $i = 1, 2$, from (12)
Compute the data vectors $\vec{v_i}$, for $i = 1, 2$, as $\vec{v_i} = D_i \cdot \vec{v_{0i}}$
Compute the coefficients $coef_i$, for $i = 1, 2$, from (8)
From $\vec{v_i}$ and $coef_i$, for $i = 1, 2$, compute $\vec{v}_i^{up}$
Normalize $\vec{v}_i^{up}$, for $i = 1, 2$ and denote it as $\{\vec{v}_i^{up}\}^*$
Construct V_i, for $i = 1, 2$
Construct the matrix M_{BI} from (17)
Compute the dominant eigenvector $\vec{\pi_{BI}}$ of the matrix M_{BI}
Compute the centrality $\vec{x}$
end

Algorithm 2: APA biplex networks algorithm to compute the node's centrality using different values of the α parameter.

Case 2: $\alpha_1 = 0.8, \alpha_2 = 0.8$ In this case, we also choose the same value of the α parameter for both layers, that is, $\alpha_i = 0.8$, for $i = 1, 2$. However, as opposed to case 1, the value of the parameter is $\alpha = 0.8$, which means that we are giving much more importance to the data than to the connectivity in the individual layers. Now, we are more interested in measuring the influence of data than the influence of the links between the nodes.

The values of the biplex centrality for this choice of α can be seen in Fig. 4 (bottom). We can see the differences with respect to the first case.

Case 3: $\alpha_1 = 0.3, \alpha_2 = 0.8$ This case under study introduces a new variant since now the values taken by the alpha parameter are different for each layer. Thus, the α_1 value for layer 1 is 0.3, which means that in layer 1 we give less importance to the data and greater to the connectivity within the network. However, we take the value $\alpha_2 = 0.8$ for layer 2, which means that we measure the importance of the nodes in this layer giving more importance to the data and less importance to the connectivity.

We can see a visualization of the biplex centrality for this case in Fig. 5 (up).

Case 4: $\alpha_1 = 0.8, \alpha_2 = 0.3$ This case is similar to case 3 with the difference that now $\alpha_1 = 0.8$ and $\alpha_2 = 0.3$. This means that we give more importance to the node attribute data in the layer 1 and to the connectivity in layer 2. The results are displayed in Fig. 5(down).

Table 1 summarizes the results of the 25 most central nodes for the four studied cases, based on the choice of α_1 and α_2 in the two layers of the network. The centrality is shown in the column labeled PGBI.

If we look, for instance, at the results obtained for case 1 and case 4, where the network and data are assigned a greater importance, respectively, we observe that of the first ten most central nodes only 2 appear in both listings: nodes 634 and 1460.

Fig. 4 offers useful information. The centrality, for the case $\alpha_1 = \alpha_2 = 0.2$ is shown in the upper part. This choice of parameters means that we consider the node attribute data as less important than the network topology in both layers. In the lower part of Fig. 4, the opposite case is shown, with $\alpha_1 = \alpha_2 = 0.8$. Now, we give much more importance to the total in-

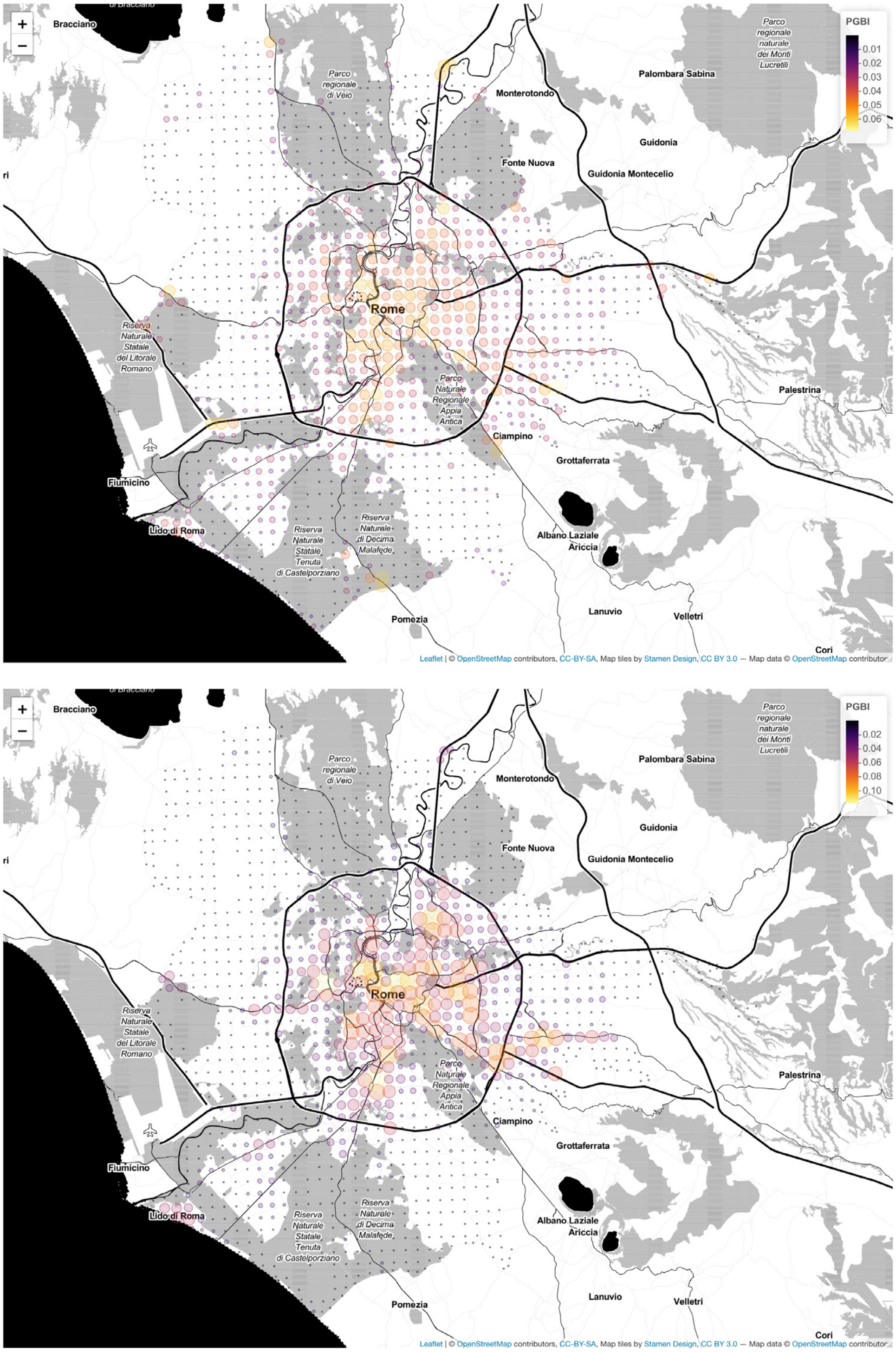


Fig. 4. Biplex centrality PGBI for $\alpha_1 = \alpha_2 = 0.2$ (top) and $\alpha_1 = \alpha_2 = 0.8$ (bottom).

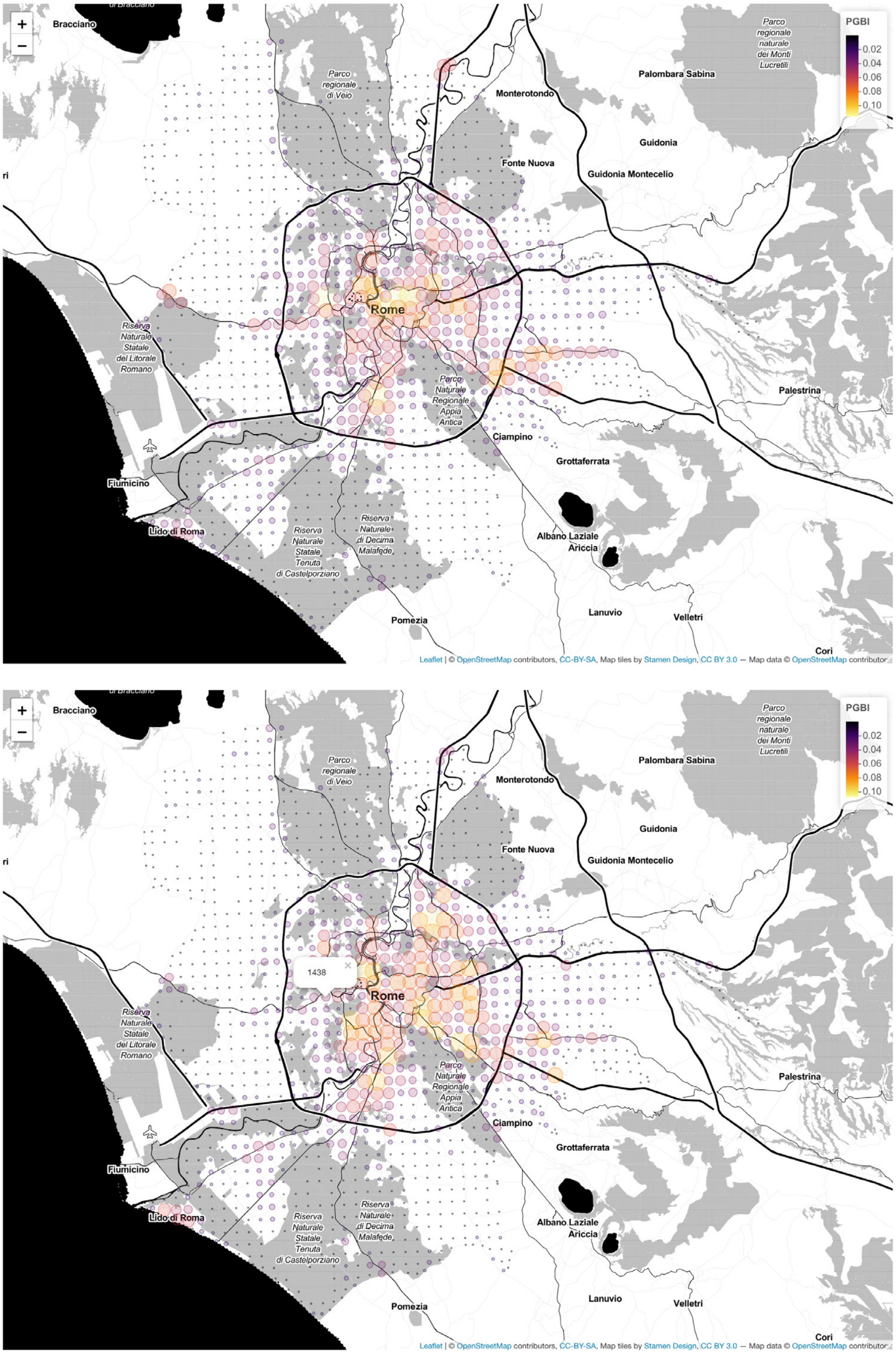


Fig. 5. Biplex centrality PGBI for $\alpha_1 = 0.3$ and $\alpha_2 = 0.8$ (top) and $\alpha_1 = 0.8$ and $\alpha_2 = 0.8$ (bottom).

Table 1
The first 25 most central nodes for the studied numerical cases.

	$\alpha_1 = 0.2,\ \alpha_2 = 0.2$		$\alpha_1 = 0.3,\ \alpha_2 = 0.8$		$\alpha_1 = 0.8,\ \alpha_2 = 0.3$		$\alpha_1 = 0.8,\ \alpha_2 = 0.8$	
ranking	node	PGBI	node	PGBI	node	PGBI	node	PGBI
1	270	0.068118	1437	0.114870	790	0.107058	1437	0.117651
2	367	0.060640	1485	0.114636	1460	0.099745	634	0.115931
3	634	0.060223	634	0.110817	776	0.096553	1485	0.112266
4	1460	0.059685	1478	0.107259	1122	0.095818	1460	0.106568
5	809	0.059439	740	0.104394	634	0.092842	790	0.104531
6	340	0.058988	1469	0.092490	673	0.092495	776	0.100668
7	44	0.058983	1460	0.091879	777	0.091264	740	0.099942
8	149	0.057718	1023	0.091119	732	0.090415	1001	0.099353
9	64	0.057618	794	0.090761	1120	0.089788	839	0.098693
10	1468	0.057595	1468	0.089183	1468	0.088741	1478	0.097365
11	150	0.057505	839	0.087562	1001	0.085205	1468	0.096991
12	301	0.057440	1013	0.086017	1437	0.083270	1469	0.093373
13	586	0.056891	776	0.083814	791	0.083016	673	0.090841
14	1469	0.055625	1486	0.082705	858	0.082981	794	0.088529
15	1122	0.055414	1014	0.082375	839	0.082880	732	0.084063
16	776	0.055333	1001	0.082046	861	0.082831	1023	0.083701
17	148	0.054751	1470	0.079746	1016	0.082021	1014	0.083121
18	1283	0.054659	1002	0.079721	860	0.080953	791	0.083013
19	740	0.054000	616	0.078429	270	0.080392	1120	0.082989
20	712	0.053724	817	0.076460	711	0.079724	858	0.082569
21	673	0.053722	738	0.076299	1469	0.078576	1013	0.082471
22	777	0.053695	1479	0.075673	712	0.078563	861	0.081994
23	714	0.053618	1438	0.074511	773	0.078317	862	0.081933
24	271	0.053441	64	0.074433	769	0.078037	777	0.080259
25	204	0.053046	739	0.073911	811	0.077536	1002	0.080107

Table 2
Spearman's correlation coefficient for the four studied cases, considering the 25 most central nodes and all the nodes, respectively.

25 nodes	$\alpha_1 = 03, \alpha_2 = 08$	$\alpha_1 = 08, \alpha_2 = 03$	$\alpha_1 = \alpha_2 = 08$	ALL nodes	$\alpha_1 = 03, \alpha_2 = 08$	$\alpha_1 = 08, \alpha_2 = 03$	$\alpha_1 = \alpha_2 = 08$
$\alpha_1 = \alpha_2 = 02$	$\rho = 0.179231$	$\rho = 0.019230$	$\rho = 0.035385$	$\alpha_1 = \alpha_2 = 02$	$\rho = 0.051641$	$\rho = 0.022459$	$\rho = 0.055890$
$\alpha_1 = 03, \alpha_2 = 08$		$\rho = 0.238461$	$\rho = 0.118461$	$\alpha_1 = 03, \alpha_2 = 08$		$\rho = 0.095983$	$\rho = 0.049532$
$\alpha_1 = 08, \alpha_2 = 03$			$\rho = 0.259231$	$\alpha_1 = 08, \alpha_2 = 03$			$\rho = 0.055624$

and out-flow as well as total bus connections associated to each node. We clearly see the difference in the maps. Specifically, in the lower figure the main roads of the city are clearly perceived, which is precisely where more public transport exists. These main roads, as well as train stations, contain nodes with a high centrality value.

In order to find whether there is a correlation between the ranking results in the four discussed cases, we compute the Spearman coefficient ρ, measuring the statistical correlation between the rankings of the nodes. A positive value of ρ near $+1$ means a high association of ranks, while a value near 0 means no association between ranks.

Table 2 shows the results of the Spearman correlation coefficient for the centralities evaluated taking the four discussed cases, for different values of α_1 and α_2. On the left part, the correlation between the 25 most central nodes in the four cases described is analyzed, while on the right all nodes are used for the calculation of the correlation. We can see that the correlation values are higher for the first 25 nodes, although in none of the cases measured the coefficient ρ is close to $+1$, which would mean an absolute correlation between the node rankings in the four cases.

This demonstrates the importance of choosing the α_i parameters in the model, giving more or less importance to the data than to the network itself.

7. Conclusions

In this paper, a measure of centrality for multiplex networks has been designed and evaluated with a real numerical example with the fundamental characteristic that both the connectivity of the graphs and a set of data present in each layer associated to the nodes are taken into account. The starting point is the original idea of the APA algorithm that introduces the influence of a set of data present in a network to the computation of the centrality of the nodes. The model solves the problem of the existence of isolated nodes in any of the layers by introducing a residual value for all nodes and representing the influence of the presence of data in the overall network. In addition, the proposed method introduces a variant with respect to the original alpha parameter related to the PageRank vector consisting of the choice of a different parameter for each of the layers. This difference in the value of α allows to take into account the importance assigned to the topology or to the data associated with the nodes in each of the layers. This allows for a flexibility that is demonstrated in the case study of the urban mobility OD and the urban bus network in the city of Rome. In that case study, a network with two layers is evaluated, where in the first layer a graph represents the OD car flows in the city, while the second layer represents local urban bus connectivity between city locations. The model solves the problem of the isolated nodes of the second layer and it allows to choose the importance of the node attribute data in each layer. Four different cases corresponding to different meaningful combinations of the α parameter are evaluated and visualised. The differences among the cases as visible from the most central city locations in each case show the advantage and utility of the proposed algorithm.

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